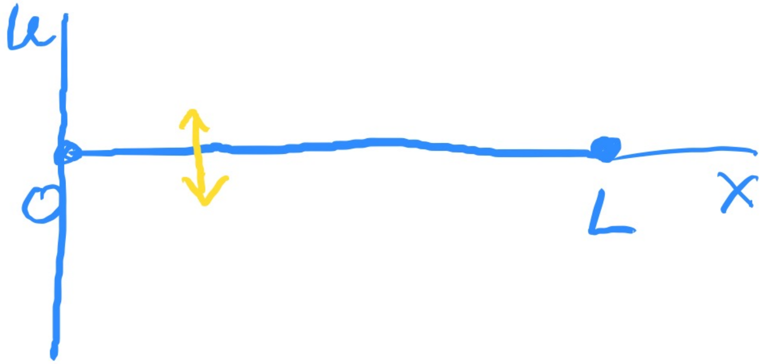
Last class: Derivation of Wave Equation

Set-up:



only vertical movement

(say string in a guitar)

calculate $F = ma$ Newton's Law

in 2 different ways for a small segment of string

① $m \approx \rho(x)\Delta x$ $\quad \rho(x) =$ mass density / length unit

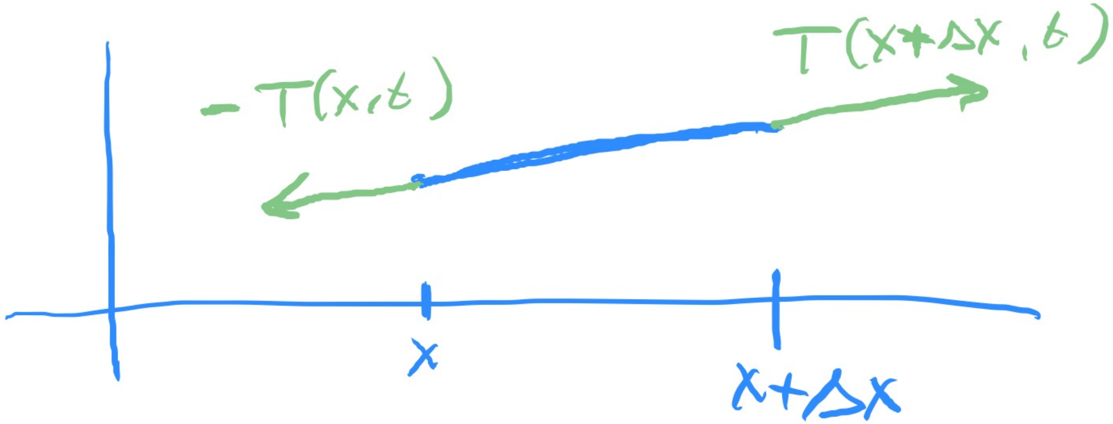
precise value: $\int_x^{x+\Delta x} \rho(y)\,dy$

$$a = \frac{\partial^2 u}{\partial t^2}(x,t)$$

$$\Rightarrow \boxed{F = \rho(x)\Delta x \,\frac{\partial^2 u}{\partial t^2}(x,t)}$$

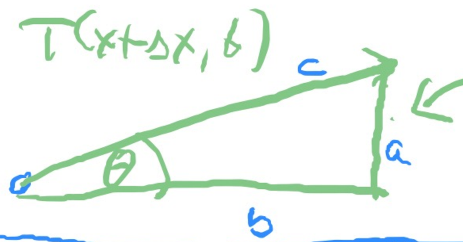
② Study forces explicitly

- forces coming from tension on string



assumptions: string elastic $\Rightarrow$ tension forces go in tangential direction

movement only in vertical direction



$T(x+\Delta x, t)\ \sin(\theta(x+\Delta x, t))$ $\boxed{\theta \text{ small}}$

$\approx$

$\tan(\theta(\ldots)) =$ slope of $u(x+\Delta x, t) =$

$= \partial/\partial x\ u(x+\Delta x, t)$

$\sin\theta = \frac{a}{c}$ $\tan\theta = \frac{a}{b}$ $c \approx b$ for small θ

$$T(x+\Delta x, t)\ \sin\theta(x+\Delta x, t)$$

$$\approx T(x+\Delta x, t)\ \frac{\partial u}{\partial x}(x+\Delta x, t)$$

Same for $T(x,t)\ \underbrace{\sin\theta(x,t)}_{\approx \frac{\partial u}{\partial x}(x,t)}$

$\Rightarrow$ force coming from tension of string

$$= \boxed{T(x+\Delta x, t)\ \frac{\partial u}{\partial x}(x+\Delta x, t) - T(x,t)\ \frac{\partial u}{\partial x}(x,t)}$$

pull in different directions

- additional forces:

  given by $Q(x,t)\, \rho(x)\, \Delta x$

  usually: $Q(x,t) = -g$ = gravity

$Q(x,t) =$ force/mass unit

get:

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}(x,t) = ma = F$$

$$= T(x+\Delta x,t)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x,t)\frac{\partial u}{\partial x}(x,t)$$

$$+ \rho(x)\Delta x \, Q(x,t)$$

$\frac{1}{\Delta x}$:

$$\Rightarrow \rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{1}{\Delta x}\left(T(x+\Delta x,t)\frac{\partial u}{\partial x}(x+\Delta x,t) - T(x,t)\frac{\partial u}{\partial x}(x,t)\right]$$

$$+ \rho(x)\,Q(x,t)$$

If $\Delta x \to 0$ get

$$\boxed{\rho(x)\frac{\partial^2 u}{\partial t^2} = \partial/\partial x\left(T(x,t)\frac{\partial u}{\partial x}(x,t)\right) + \rho(x)\,Q(x,t)}$$

Simplifications:

Assume ς and T to be constant

$$\Rightarrow \quad \boxed{\varsigma \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} + \varsigma Q(x,t)}$$

equilibrium state: $\frac{\partial^2 u}{\partial t^2} = 0$

$$\Rightarrow \text{ get ODE } \quad \boxed{T \frac{\partial^2 u}{\partial x^2} + \varsigma Q(x,t) = 0}$$

Often Q is negligible compared to T

$\Rightarrow$ get one-dimensional wave equation

$$\varsigma \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\varsigma}}$$

4.4 Wave equation with fixed boundaries

Problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(PDE)}$$

$$u(0,t) = 0 = u(L,t) \qquad \text{(BC)}$$

$$u(x,0) = f(x) \qquad \text{(IC)}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Remark: have second order DE with respect to t

$\Rightarrow$ need two initial conditions

We can solve this by same methods as for heat equation!

- consider product solutions $u(x,t) = \phi(x)\, h(t)$
- plug into PDE $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\rightarrow \quad \phi(x)\, h''(t) = c^2 \phi''(x)\, h(t) \qquad \frac{1}{c^2 \phi(x) h(t)}$$

$$\rightarrow \quad \boxed{\frac{h''(t)}{c^2 h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda}$$

(BC): $u(0,t) = 0 = u(L,t) \Rightarrow \boxed{\phi(0) = 0 = \phi(L)}$

$$\boxed{\phi''(x) = -\lambda\, \phi(x)}$$

as before get:

$$\boxed{\phi(x) = \sin \frac{n\pi}{L} x, \qquad n = 1, 2, \ldots \qquad \lambda = \left(\frac{n\pi}{L}\right)^2}$$

similarly: $h'' = -\lambda c^2 h$

$$= -\left(\frac{n\pi c}{L}\right)^2 h \qquad , n = 1, 2 \ldots$$

$$\Rightarrow \boxed{h(x) = A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t}$$

$\Rightarrow$ get series solution

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi c}{L} t \sin \frac{n\pi}{L} x + B_n \sin \frac{n\pi c}{L} t \sin \frac{n\pi}{L} x}$$

Calculate Fourier coefficients using initial conditions

$$\boxed{u(x,0) = f(x) = \sum A_n \sin \frac{n\pi}{L} x}$$

$$\left(\cos \frac{n\pi}{L} 0 = 1\right)$$

$$\Rightarrow \boxed{A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi}{L} \sin \frac{n\pi}{L} x$$

$$\Rightarrow \boxed{B_n \frac{n\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx}$$